$\mathcal{PT}$-symmetric State Discrimination

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Contents

1 Introduction

1.1 $\mathcal{PT}$-symmetric quantum mechanics

The role of the Hamiltonian H in a quantum-mechanical system is to determine the possible values of the system's energy, which are the eigenvalues of H, as well as to dictate its evolution in time via the Schrödinger equation, $i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$. In order to describe a sensible physical system, both of these imply something about the most general H that might be considered. Because we only observe systems with real energies, the former means that all its eigenvalues must be real. In order to guarantee conservation of probability, the latter implies that the time-evolution operator, which for time-independent Hamiltonians may be written as e^{-iHt}, must be unitary.

The usual postulates of quantum mechanics include the Hermiticity of the Hamiltonian, $H = H^\dagger$, with respect to the canonical inner product on $\mathbb{C}^n$. This is a sufficient but not necessary condition to ensure the reality of its eigenvalues. However, since an inner product on a complex Hilbert space is *sesquilinear* (linear in one argument and linear with complex conjugation in the other), unitarity of the time-evolution operator implies that H must be self-adjoint with respect to the inner product that we consider.

Since 1998, when it was discovered (first by numerical and asymptotic methods [1], and only in 2001 shown exactly [2]) that the family of Hamiltonians given by

$$H = \hat{p}^2 + \hat{x}^2(i\hat{x})^\epsilon \tag{1}$$

has real eigenvalues if $\epsilon \geq 0$, $\mathcal{PT}$ symmetry has been brought forward as a possible generalisation for Hermiticity, where $\mathcal{P}$ and $\mathcal{T}$ are, respectively, the parity and time-reversal operators, and we say that H is $\mathcal{PT}$-symmetric if $[H, \mathcal{PT}] = 0$. In general, a $\mathcal{PT}$-symmetric Hamiltonian has real eigenvalues if H and $\mathcal{PT}$ have the same

eigenvectors. Note that even though they commute, they do not always have the same eigenvectors because $\mathcal{PT}$ is an antilinear and not a linear operator. If H and $\mathcal{PT}$ do not have the same eigenvectors, we say that the $\mathcal{PT}$ symmetry is *broken*. The Hamiltonians defined in (1), for example, are in a parametric region of broken $\mathcal{PT}$ symmetry for $\epsilon < 0$, while $\mathcal{PT}$ symmetry is unbroken at $\epsilon \geq 0$.

It has since then been shown that if a given Hamiltonian has unbroken $\mathcal{PT}$ symmetry, it is also possible to construct a positive-definite inner product for which H is self-adjoint, and hence e^{-iHt} is unitary. This construction involves finding a new symmetry of the Hamiltonian, represented by a linear operator $\mathcal{C}$, which only exists in the region of unbroken $\mathcal{PT}$ symmetry. The essential ingredients leading to $\mathcal{C}$ will be outlined in section 2.2.

1.2   Two related problems

Two interesting problems in quantum mechanics that are also easy to formulate are:

Problem 1: the quantum brachistochrone

Given two states $|\psi_I\rangle$ and $|\psi_F\rangle$, can we find a Hamiltonian H under which $|\psi_I\rangle$ evolves into $|\psi_F\rangle$? Can this evolution be arbitrarily fast?

Problem 2: state discrimination

Given one quantum system which we know is in one of two states $|\psi\rangle$ and $|\phi\rangle$, how can we tell which state the system is in? Is it possible to be sure?

These two problems are similar because the relevant subspace of the Hilbert space is two-dimensional for both problems. For the brachistochrone, the shortest path between the states doesn't leave the subspace that they span. In state discrimination, the important variable is the separation between the two states, determined by their inner product. A two-dimensional space in quantum mechanics can be represented by the Bloch sphere, a sphere where each point represents a unique state

and the overlap is larger for pairs of states that are closer (we will define the Bloch sphere in more detail in section 3).

In conventional quantum mechanics, one finds that the speed of evolution is proportional to the energy variance, which for two-dimensional systems depends only on the difference between the largest and smallest eigenvalues of H. Hence, if we limit this difference, which is limited in any practical application, a maximum speed and a minimum evolution time are obtained for the quantum brachistochrone. If we allow for $\mathcal{PT}$-symmetric Hamiltonians, however, this limit does not exist. Even if we limit the difference between energy eigenvalues, a given state can evolve into any other state in an arbitrarily short time!

As for state discrimination, unless the states are orthogonal, one measurement is not sufficient to distinguish them with certainty, and after we make one measurement the system is no longer in its original state. If $|\langle\phi|\psi\rangle|$ is large (close to 1), even if we have access to many systems prepared in the same state, the best that can be done is to devise a measurement strategy that minimizes the probability of making a mistake.

Since in the $\mathcal{PT}$-symmetric theory the inner product is determined by the Hamiltonian, a possible new strategy to solve this problem would be to find an inner product for which $|\psi\rangle$ and $|\phi\rangle$ are orthogonal. We will show that this strategy does indeed work, and that state discrimination is also simplified immensely by the use of $\mathcal{PT}$-symmetric Hamiltonians.

In this dissertation, we will start by summarizing the main features of $\mathcal{PT}$-symmetric quantum mechanics. We will then review the conventional theory on the quantum brachistochrone, as well as the work of Bender *et al.* [3], who discovered that the use of $\mathcal{PT}$-symmetric Hamiltonians allows for arbitrarily fast evolutions. This is followed by a short introduction to the problem of state discrimination and how it would be possible to solve it using the framework of $\mathcal{PT}$ symmetry.

2 $\mathcal{PT}$ Symmetry

In order to define $\mathcal{PT}$ symmetry properly, we need to define the operators $\mathcal{P}$ and $\mathcal{T}$. The parity operator, $\mathcal{P}$, corresponds logically to a reflection of the spatial coordinates. This means that $\mathcal{P}$ is a linear operator that changes the sign of both the position and the momentum operators,

$$\mathcal{P}\hat{x}\mathcal{P} = -\hat{x}, \quad \mathcal{P}\hat{p}\mathcal{P} = -\hat{p}. \tag{2}$$

The time-reversal operator corresponds to a reflection of the time coordinate, leaving space unchanged. This implies that $\mathcal{T}$ changes the sign of the momentum operator,

$$\mathcal{T}\hat{x}\mathcal{T} = \hat{x}, \quad \mathcal{T}\hat{p}\mathcal{T} = -\hat{p}. \tag{3}$$

In order to keep the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar\mathbf{1}$ invariant, $\mathcal{T}$ must be antilinear, that is, $\mathcal{T}$ must perform complex conjugation when acting on scalars: $\mathcal{T}i\mathcal{T} = -i$. Both $\mathcal{P}$ and $\mathcal{T}$ are reflection operators, so applying any of them twice should leave a system unchanged, and since they reflect different coordinates, the action of one should not affect the other. This implies that $\mathcal{P}^2 = \mathcal{T}^2 = \mathbf{1}$ and that $[\mathcal{P}, \mathcal{T}] = 0$, which together imply that $(\mathcal{PT})^2 = \mathbf{1}$.

A Hamiltonian H is then said to be $\mathcal{PT}$-symmetric if $[H, \mathcal{PT}] = 0$, or equivalently, if

$$H^{\mathcal{PT}} := (\mathcal{PT})H(\mathcal{PT}) = H. \tag{4}$$

If (4) holds, the energy eigenvalues are real if and only if $\mathcal{PT}$ symmetry is unbroken, that is, iff H and $\mathcal{PT}$ have the same eigenvectors. To show this, suppose that H is diagonalisable with a discrete spectrum. Since $H \neq H^{\dagger}$, both H and its adjoint

have complete sets of eigenvectors:

$$H|E_n\rangle = E_n|E_n\rangle, \quad H^\dagger|E^m\rangle = E^m|E^m\rangle, \tag{5}$$

where in general $E^n = E_n^*$. By the definition of adjoint we know

$$\langle E^m|H|E_n\rangle = \langle E_n|H^\dagger|E^m\rangle^* \Rightarrow (E_n - E_m)\langle E^m|E_n\rangle = 0 \Rightarrow \langle E^m|E_n\rangle = \delta_{mn}. \tag{6}$$

We also have $\langle E_n|E^m\rangle = \delta_{mn}$, and

$$\sum_n |E^n\rangle\langle E_n| = \sum_n |E_n\rangle\langle E^n| = \mathbf{1}. \tag{7}$$

Let us first assume that H has unbroken $\mathcal{PT}$ symmetry. Let λ_n be the eigenvalue of $\mathcal{PT}$ corresponding to $|E_n\rangle$:

$$H|E_n\rangle = E_n|E_n\rangle, \quad \mathcal{PT}|E_n\rangle = \lambda_n|E_n\rangle. \tag{8}$$

Multiply both the equations by $\mathcal{PT}$ on the left, and insert $(\mathcal{PT})^2 = \mathbf{1}$, to obtain

$$(\mathcal{PT})H|E_n\rangle = (\mathcal{PT})E_n(\mathcal{PT})^2|E_n\rangle, \quad |E_n\rangle = (\mathcal{PT})\lambda_n(\mathcal{PT})^2|E_n\rangle. \tag{9}$$

Using the fact that H and $\mathcal{PT}$ commute, and that $\mathcal{PT}$ is antilinear, we have

$$H\lambda_n|E_n\rangle = E_n^*\lambda_n|E_n\rangle, \quad |E_n\rangle = \lambda_n^*\lambda_n|E_n\rangle. \tag{10}$$

The second equation tells us that $|\lambda_n|^2 = 1$, so we can use the linearity of H to divide by λ_n in the first equation and get $E_n = E_n^*$, which implies that E_n is indeed real.

To prove the converse, now suppose that E_n is real. We can multiply the first

equation in (5) by $\mathcal{PT}$ and use $[H, \mathcal{PT}] = 0$, obtaining

$$H\mathcal{PT}|E_n\rangle = E_n^*\mathcal{PT}|E_n\rangle = E_n\mathcal{PT}|E_n\rangle. \tag{11}$$

If we then multiply by $\langle E^m|$ on the left, doing the same manipulation as in (6), we obtain

$$(E_n - E_m)\langle E^m|\mathcal{PT}|E_n\rangle = 0. \tag{12}$$

This implies that $\mathcal{PT}|E_n\rangle$ is a multiple of $|E_n\rangle$, so $|E_n\rangle$ is also an eigenvector of $\mathcal{PT}$.

2.1   Boundary conditions

When solving the Schrödinger eigenvalue problem set by a given Hamiltonian, $H\psi = E\psi$, the standard approach is to write the Hamiltonian in the coordinate representation. But to solve the resulting differential equation, we also need to specify some boundary conditions for the eigenfunctions. Usually, we require that the wave function be square-integrable along the real axis. Here, however, in order to get a theory with unbroken $\mathcal{PT}$ symmetry, we have to make sure that the eigenfunctions of the Hamiltonian are also eigenfunctions of $\mathcal{PT}$. Thus we must treat the coordinates as complex numbers, so we can use $\mathcal{PT}$-symmetric boundary conditions. Note that since $\mathcal{P} : x \to -x$ and $\mathcal{T} : x \to x^*$, this means that the boundary conditions must be symmetric under $x \to -x^*$, or reflection through the imaginary axis.

A possible way to identify the appropriate boundary conditions for a family of Hamiltonians, such as (1), is to make use of a well-known Hermitian case, which for (1) is the harmonic oscillator, i.e., $\epsilon = 0$. Writing the Schrödinger equation in the coordinate representation, where

$$\hat{x} \to x, \quad \hat{p} \to -i\frac{d}{dx}, \tag{13}$$

we find

$$-\psi''(x) + x^2(ix)^\epsilon \psi(x) = E\psi(x). \tag{14}$$

To understand which boundary conditions we should impose in the complex-x plane, all we need to know is the asymptotic behaviour of $\psi(x)$ as $|x| \to \infty$. To find the asymptotic behaviour we can use the WKB approximation. This tells us that if $y(x)$ is a solution to a differential equation of the form $-y''(x) + V(x)y(x) = 0$, where $V(x)$ grows as $|x| \to \infty$, then the controlling factor of the asymptotic behaviour of $y(x)$ is an exponential of the form

$$\exp\left(\pm \int^x du\sqrt{V(u)}\right). \tag{15}$$

For the case of the harmonic oscillator wave function, this becomes $\exp\left(\pm\frac{1}{2}x^2\right)$, so requiring that the wave function be square-integrable along the real axis is equivalent to choosing the negative sign, or, if we write $x = re^{i\theta}$, requiring that $\psi(x)$ go to 0 when $r \to \infty$ for the angles $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$. The wedges on the complex plane where we require the eigenfunction to vanish asymptotically are called *Stokes wedges* [4].

Now we proceed by smoothly continuing these wedges for the case $\epsilon > 0$. Here, one finds that the exponential factor of $\psi(x)$ as $|x| \to \infty$ is $\exp\left(\pm\frac{2}{\epsilon+4}i^{\frac{\epsilon}{2}}x^{\frac{\epsilon+4}{2}}\right)$. To have our solution be a continuation of the harmonic oscillator, we want to keep choosing the negative sign solution. To find the wedges where this solution vanishes asymptotically, we need to solve for the condition

$$\mathrm{Re}\left(i^{\frac{\epsilon}{2}}x^{\frac{\epsilon+4}{2}}\right) > 0 \Leftrightarrow \cos\left(\frac{\epsilon+4}{2}\theta + \frac{\pi\epsilon}{4}\right) > 0$$
$$\Leftrightarrow \frac{-2-\epsilon+8k}{\epsilon+4}\frac{\pi}{2} < \theta < \frac{2-\epsilon+8k}{\epsilon+4}\frac{\pi}{2}. \tag{16}$$

For $\epsilon = 0$, any even k gives an equivalent condition on x, because the substitution

of k by $k+2$ simply adds 2π to the solutions for θ. For the same reason, any odd k also gives an equivalent condition. Thus, we just need to take one of each to arrive at the conditions we found for the harmonic oscillator.

If we choose the same values of k for any ϵ, the wedges move smoothly with ϵ. We then take the even k to be 0, and in order to arrive at $\mathcal{PT}$-symmetric boundary conditions for arbitrary ϵ we need to take -1 as the odd k. This means that we require the wave function to vanish at infinity in two wedges of width $\frac{2\pi}{\epsilon+4}$ centred around $\theta = -\frac{\epsilon}{\epsilon+4}\,\frac{\pi}{2}$ and $\theta = \pi + \frac{\epsilon}{\epsilon+4}\,\frac{\pi}{2}$.

Note that these wedges are not invariant under complex conjugation (reflection through the real axis) for $\epsilon \neq 0$, which means that even when the Hamiltonian operator by itself looks Hermitian (which happens for even-integer values of ϵ), it is non-Hermitian in this context. In fact, for $\epsilon \geq 2$, the wedges do not even include the real axis, so the wave functions are only square-integrable along complex contours that go to infinity inside the Stokes wedges.

This is the sense in which this theory is often called a "complex extension" of quantum mechanics [5, 6].

2.2   Inner product and $\mathcal{C}$ operator

The other essential ingredient of this theory, as we have anticipated in section 1.1, is the construction of a positive-definite inner product with respect to which H is self-adjoint, so that the time-evolution operator e^{-iHt} is unitary and we obtain a consistent probabilistic interpretation. Since $[H, \mathcal{PT}] = 0$ and $\mathcal{PT}$ is antilinear, a logical first attempt to construct an inner product would be acting with $\mathcal{PT}$ on one of the vectors:

$$(\phi, \psi) := (\mathcal{PT}|\phi\rangle) \cdot |\psi\rangle = \int_C dx\, [\phi(-x)]^*\psi(x), \qquad (17)$$

where C is a complex contour that goes to infinity inside the Stokes wedges as described above. However, it turns out that this candidate to inner product is not

positive-definite, leading to states with negative norm, unacceptable in a quantum theory (we will explicitly solve one example with negative norm states in section 3.2).

The most important step towards finding this positive-definite inner product is thus to find a new linear operator, which we call $\mathcal{C}$ [5]. This operator must commute with both H and $\mathcal{PT}$, and be such that $(\mathcal{CPT}|\phi\rangle) \cdot |\psi\rangle$ is a positive-definite inner product. Note that H is automatically self-adjoint with respect to this new inner product, since H commutes with $\mathcal{CPT}$ because it commutes with both $\mathcal{C}$ and $\mathcal{PT}$.

We begin by normalising the simultaneous eigenfunctions of H and $\mathcal{PT}$ such that the eigenvalue of $\mathcal{PT}$ is 1. We have shown in (10) that for any eigenfunction ψ_n, $\mathcal{PT}\psi_n = e^{i\alpha_n}\psi_n$. Hence, if we redefine the eigenfunctions as $\phi_n = e^{-i\alpha_n/2}\psi_n$, the new eigenfunctions satisfy

$$H\phi_n(x) = E_n\phi_n(x), \quad \mathcal{PT}\phi_n(x) = \phi_n(x). \tag{18}$$

We can then show [7] that

$$\int_C dx[\phi_n(-x)]^*\phi_n(x) = s_n, \quad s_n = \pm 1, \tag{19}$$

that is, the $\mathcal{PT}$ norms of the eigenfunctions take the values ± 1. Moreover, the eigenfunctions corresponding to different energy eigenvalues are orthogonal with respect to this inner product, and we have a modified statement of completeness:

$$\sum_n s_n\phi_n(x)\phi_n(y) = \delta(x - y). \tag{20}$$

Proof of (19) **and** (20)   To show this, we will suppose that $H^\dagger = \mathcal{P}H\mathcal{P}$. This is equivalent to $\mathcal{PT}$ symmetry if $H(\hat{x}, \hat{p})$ only has even-integer powers of $\hat{p}$, which is true for most physically interesting cases, where $H = \hat{p}^2 + V(\hat{x})$. In matrix form,

$H^\dagger = \mathcal{P}H\mathcal{P}$ is equivalent to $[H, \mathcal{PT}] = 0$ if H is symmetric under transposition.

Recovering the second equation of (5) and using $H^\dagger = \mathcal{P}H\mathcal{P}$, we obtain $\mathcal{P}H\mathcal{P}|E^n\rangle = E_n|E^n\rangle$, where we used that E_n is real, so $E^n = E_n$. Since $\mathcal{P}^2 = \mathbf{1}$, we now multiply by $\mathcal{P}$ to get $H\mathcal{P}|E^n\rangle = E_n\mathcal{P}|E^n\rangle$. This is similar to (11), and implies that

$$\mathcal{P}|E^n\rangle = s_n|E_n\rangle \Leftrightarrow |E^n\rangle = s_n\mathcal{P}|E_n\rangle, \tag{21}$$

where $s_n \in \mathbb{C}$. We can then show that we can choose $s_n = \pm 1$. We start by showing that it is real:

$$1 = \langle E_n|E^n\rangle = \langle E_n|\mathcal{P}^2|E^n\rangle = s_n^{*-1}s_n\langle E^n|E_n\rangle = s_n^{*-1}s_n, \tag{22}$$

thus $s_n = s_n^*$. Multiplying each side of (21) by its adjoint and using $\mathcal{P}^2 = \mathbf{1}$, we then have

$$s_n^2 = \frac{\langle E^n|E^n\rangle}{\langle E_n|E_n\rangle}. \tag{23}$$

This can be made equal to one by rescaling the bases while keeping the normalisation. This means rescaling $|E_n\rangle \to \lambda_n|E_n\rangle$ and $|E^n\rangle \to \lambda_n^{-1}|E^n\rangle$, keeping $\langle E^m|E_n\rangle = \delta_{mn}$ invariant. Then $s_n^2 \to \lambda_n^{-4}s_n^2$, and since can choose λ_n to be any real number, we choose it such that $s_n^2 = 1 \Leftrightarrow s_n = \pm 1$.

The second equality in (7) can then be written, in the coordinate representation,

$$\sum_n |E_n\rangle\langle E_n|s_n\mathcal{P} = \mathbf{1} \Leftrightarrow \sum_n s_n\phi_n(x)\phi_n^*(-y) = \sum_n s_n\phi_n(x)\phi_n(y) = \delta(x-y), \tag{24}$$

where we used $\mathcal{PT}\phi_n(x) = \phi_n^*(-x) = \phi_n(x)$, from the way we defined the eigenfunctions ϕ_n. Similarly, the orthogonality relations can be written

$$\delta_{nm} = \langle E_m|E^n\rangle = s_n\langle E_m|\mathcal{P}|E_n\rangle = s_n\int_C dx\langle E_m|\mathcal{P}|x\rangle\langle x|E_n\rangle. \tag{25}$$

Note that in the coordinate representation, the integral is made along a complex contour inside the Stokes wedges, as defined above. We then have

$$\int_C dx\, \phi_m^*(-x)\phi_n(x) = s_n\delta_{mn}, \tag{26}$$

which is what we wanted. Using $\mathcal{PT}\phi_n(x) = \phi_n(x)$, we can also write

$$\int_C dx\, \phi_m(x)\phi_n(x) = s_n\delta_{mn}. \tag{27}$$

$$\square$$

The operator $\mathcal{C}$ is then defined in the coordinate representation as

$$\mathcal{C}(x,y) = \sum_n \phi_n(x)\phi_n(y). \tag{28}$$

We can then verify that $\mathcal{C}^2 = 1$ and $\mathcal{C}$ commutes with H and $\mathcal{PT}$. Also, $\mathcal{C}\phi_n(x) = s_n\phi_n(x)$, so the in the new inner product all eigenfunctions have norm 1:

$$(\mathcal{CPT}|\phi_n\rangle) \cdot |\phi_n\rangle = s_n^2 = 1. \tag{29}$$

Proof   In this proof we will use the coordinate representation and equations (20) and (27) throughout. Let us begin by $\mathcal{C}^2 = 1$:

$$\langle x|\mathcal{C}^2|z\rangle = \int dy\, \mathcal{C}(x,y)\mathcal{C}(y,z) = \sum_n \sum_m \int dy\, \phi_n(x)\phi_n(y)\phi_m(y)\phi_m(z)$$

$$= \sum_n \sum_m \phi_n(x)s_n\delta_{mn}\phi_m(z) = \sum_n s_n\phi_n(x)\phi_n(z)$$

$$= \delta(x-z) = \langle x|z\rangle. \tag{30}$$

To check that $[\mathcal{C}, H] = [\mathcal{C}, \mathcal{PT}] = 0$, it is enough to prove that $\mathcal{C}\phi_n(x) = s_n\phi_n(x)$, because the existence of a complete set of mutual eigenvectors with real eigenvalues

for all three operators proves that they commute. We then calculate

$$\int dy \mathcal{C}(x,y)\phi_n(y) = \int dy \sum_m \phi_m(x)\phi_m(y)\phi_n(y) = \sum_m \phi_m(x)s_n\delta_{mn} = s_n\phi_n(x).$$
(31)

Hence,

$$(\mathcal{CPT}|\phi_n\rangle) \cdot |\phi_n\rangle = s_n(\mathcal{PT}|\phi_n\rangle) \cdot |\phi_n\rangle = s_n^2 = 1.$$
(32)

□

For the family of Hamiltonians in (1), we have the special case $s_n = (-1)^n$. When calculating $\mathcal{C}$ for a general Hamiltonian, what is done in practice [8] is to solve for the 3 conditions that $\mathcal{C}$ must satisfy,

$$\mathcal{C}^2 = \mathbf{1}, \quad [\mathcal{C}, \mathcal{PT}] = 0, \quad [\mathcal{C}, H] = 0.$$
(33)

An ansatz that is usually followed when performing this calculation is

$$\mathcal{C} = e^{Q(\hat{x},\hat{p})}\mathcal{P},$$
(34)

which implies that $\mathcal{CPT} = e^Q\mathcal{T}$. This ansatz does not work if H is not diagonalisable, though the operator $\mathcal{C}$ still exists.

2.3 Pseudo-Hermiticity and Hermitian equivalent

Mostafazadeh [9] pointed out that the condition that a Hamiltonian H be $\mathcal{PT}$-symmetric can be understood more generally as a special case of pseudo-Hermiticity. An operator H is said to be *pseudo-Hermitian* if

$$H^\dagger = \eta H \eta,$$
(35)

where η is Hermitian. If $\eta = \mathcal{P}$, this condition is the same as $\mathcal{PT}$ symmetry for transposition-symmetric H, as we have seen below equation (20).

Note that e^Q in (34) is a *positive operator* (a self-adjoint operator with nonnegative real eigenvalues), since we must be able to write any eigenvalue of e^Q as e^q, where q is an eigenvalue of Q. For any positive operator T, there is a unique positive operator A such that $A^2 = T$ [10]. Making use of the fact that e^Q is positive, Mostafazadeh [11] has shown that if (34) holds, there is a Hermitian theory equivalent to the $\mathcal{PT}$-symmetric theory. If we map the $\mathcal{PT}$-symmetric Hamiltonian H to

$$h = e^{-Q/2}He^{Q/2}, \tag{36}$$

where $e^{Q/2}$ is the operator that squares to e^Q, then h and H have the same eigenvalues and it can be shown that h is Hermitian:

$$h^\dagger = e^{Q/2}H^\dagger e^{-Q/2}, \tag{37}$$

which, if we note that $e^Q = e^Q\mathcal{PP} = \mathcal{CP}$ and $e^{-Q} = (\mathcal{CP})^{-1} = \mathcal{PC}$, can also be written as

$$h^\dagger = e^{-Q/2}\mathcal{CP}H^\dagger\mathcal{PC}e^{Q/2} = e^{-Q/2}He^{Q/2} = h, \tag{38}$$

where we used $H^\dagger = \mathcal{PHP}$ and $[\mathcal{C}, H] = 0$.

If the states in the Hilbert space are also mapped accordingly, $|\psi\rangle \rightarrow e^{Q/2}|\psi\rangle$, this is a unitary similarity transformation between the system described by the $\mathcal{PT}$-symmetric Hamiltonian with the $\mathcal{CPT}$ inner product and the system described by h with the canonical inner product. Thus any system described by a diagonalisable $\mathcal{PT}$-symmetric Hamiltonian can also be fully described in the context of usual Hermitian Hamiltonians.

One could then think that allowing for $\mathcal{PT}$-symmetric Hamiltonians is not a real generalisation, since we can simply work with the Hermitian equivalent. Note,

however, that $e^{Q/2}$ is not unitary if used to map between two Hilbert spaces with the canonical inner product. This means that we are not allowed to map directly from the usual Hilbert space of quantum mechanics to the Hermitian equivalent to a $\mathcal{PT}$-symmetric theory. Hence, the replacement of Hermiticity by $\mathcal{PT}$ symmetry can still be an effective generalisation, at least in terms of allowing for different inner products on the vector space used to describe a physical system.

Furthermore, there is also a difficulty of a practical nature in utilising this map that takes a $\mathcal{PT}$-symmetric Hamiltonian to a Hermitian one: the form of the transformation is terribly complicated, except for a few special known cases. It is usually only possible perturbatively, and the resulting Hermitian Hamiltonian h has arbitrarily high powers of $\hat{x}$ and $\hat{p}$ [12], and is thus nonlocal.

2.4   Experimental breakthroughs

Until recently, no one had yet found a way to experimentally test the concepts of $\mathcal{PT}$ symmetry in quantum-mechanical systems. Nevertheless, some optical processes such as spatial diffraction and temporal dispersion obey differential equations that are formally equivalent to the Schrödinger equation. Systems obeying such equations can thus be considered as possible simulators for some concepts of $\mathcal{PT}$ symmetry [13, 14]. What plays the role of the potential $V(x)$ in these equations is the complex refractive-index distribution, $n(x) = n_R(x) + i n_I(x)$.

For a simple Hamiltonian of the form $H = \frac{p^2}{2m} + V(x)$, the condition of $\mathcal{PT}$ symmetry is equivalent to $V(x) = V^*(-x)$. Hence, the conditions we need to impose on the real and imaginary parts of the refractive index are:

$$n_R(x) = n_R(-x), \quad n_I(x) = -n_I(-x), \tag{39}$$

which can be implemented through index guiding and careful balancing of the gain and loss regions.

The first feature of $\mathcal{PT}$-symmetric Hamiltonians that was successfully tested was $\mathcal{PT}$-symmetry breaking [15, 16]. It was observed that, as predicted, when the parameters passed a given threshold the eigenmodes ceased to have $\mathcal{PT}$ symmetry and the corresponding eigenvalues started to have a nonzero imaginary part. Further applications of these concepts have been predicted [17], and more experimental results are expected to appear in the forthcoming years. These findings also have great importance in optics and photonics, as the so-called $\mathcal{PT}$ *materials* (materials with $\mathcal{PT}$-symmetric refractive-index distribution) are anticipated to have various applications in those fields.

The diffusion equation can also be considered in a form that makes it equivalent to the Schrödinger equation. In this context, some effects of $\mathcal{PT}$ symmetry have also been observed in the diffusion of spin-polarized rubidium atoms [18].

Very recently, a description of transport in a superconducting weak link has been made in terms of $\mathcal{PT}$-symmetric Hamiltonians [19]. If this is confirmed, the first quantum-mechanical system described by a $\mathcal{PT}$-symmetric Hamiltonian has been found. This would be a very interesting development towards a possible experimental test of all these concepts in quantum mechanics.

3 Quantum Brachistochrone

The classical problem of finding the shape of the minimal-time trajectory between two given points, for a particle under the sole action of gravity, is usually called the brachistochrone problem, from the Greek *brachistos* (shortest) and *chronos* (time). An interesting problem in quantum mechanics, coined the "quantum brachistochrone" by Carlini *et al.* [20], is the one of finding the Hamiltonian under which the initial state of the system $|\psi_I\rangle$ evolves into a given final state $|\psi_F\rangle$ in the least possible time. This problem has applications in quantum computing, where it is related to the processing time, among other areas.

The shortest path between two states lies in the two-dimensional subspace spanned by them, so we can simplify the problem by working in that smaller space. Since two vectors related by a scalar multiplication represent the same state, the space that describes quantum mechanics in n dimensions is not $\mathbb{C}^n$, but $\mathbb{C}^n/\sim$, where $\sim$ is the equivalence relation $|\psi\rangle \sim \lambda|\psi\rangle$ if $\lambda \in \mathbb{C}$. $\mathbb{C}^n/\sim$ is usually known as $\mathbb{C}P^{n-1}$, the $(n-1)$-dimensional complex projective space. For $n = 2$, we have $\mathbb{C}P^1$, which is diffeomorphic to S^2, the 2-sphere. Furthermore, the usual metric on $\mathbb{C}P^n$, the *Fubini-Study metric*, applied to $\mathbb{C}P^1$ is equivalent to the natural metric on S^2. The correspondence commonly used between $\mathbb{C}P^1$ and S^2 is

$$(\theta, \varphi) \leftrightarrow \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix}, \tag{40}$$

where θ and φ are spherical coordinates on S^2, and we choose the vector in $\mathbb{C}^2$ to be normalised and adjust the phase so that the first element is real.

Each state in 2-dimensional quantum mechanics can then be represented by a point on a 2-sphere, called the *Bloch sphere*. If we consider the points on the Bloch sphere corresponding to two states $|\psi_I\rangle$ and $|\psi_F\rangle$, the angular distance between them depends only on the inner product $\langle\psi_F|\psi_I\rangle$, being given by $\alpha = 2\arccos|\langle\psi_F|\psi_I\rangle|$. In particular, two orthogonal states are always represented by antipodal points on the sphere. We have then shown that the question of how fast $|\psi_I\rangle$ can evolve to $|\psi_F\rangle$ is simply the question of how fast states can move on the Bloch sphere.

3.1   In conventional quantum mechanics

In 1990 [21], Anandan and Aharonov proved what is now known as the Anandan-Aharonov relation, which states that the speed of evolution of a quantum system by

a Hermitian Hamiltonian is given by

$$\frac{ds}{dt} = \frac{2\Delta E}{\hbar},\tag{41}$$

where ds is the distance on the complex projective space given by the Fubini-Study metric, and $(\Delta E)^2$ is the energy variance, defined as $(\Delta E)^2 := \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2$. Note that for time-independent Hamiltonians ΔE is constant, since the time-evolution operator is unitary and commutes with H.

We can then see that the limiting factor of ΔE on a two-dimensional system is the difference between the largest and smallest eigenvalues [22]. Since the Hamiltonian is diagonalisable, we can write $H = \lambda_+|E_+\rangle\langle E_+| + \lambda_-|E_-\rangle\langle E_-|$, and a general state as $|\psi\rangle = a|E_+\rangle + b|E_-\rangle$ with $|a|^2 + |b|^2 = 1$. Thus

$$\begin{aligned}(\Delta E)^2 &= |a|^2\lambda_+^2 + |b|^2\lambda_-^2 - (|a|^2\lambda_+ + |b|^2\lambda_-)^2 \\ &= |a|^2|b|^2(\lambda_+ - \lambda_-)^2.\end{aligned}\tag{42}$$

We can orient the Bloch sphere such that $|E_+\rangle$ lies at the North Pole ($\theta = 0$) and $|E_-\rangle$ at the South Pole ($\theta = \pi$). Then $|a| = \cos\frac{\theta}{2}$ and $|b| = \sin\frac{\theta}{2}$, so

$$\Delta E = \frac{1}{2}\sin\theta|\lambda_+ - \lambda_-|.\tag{43}$$

The factor of $\sin\theta$ can be easily interpreted. Since time evolution is unitary, the map from the original to the evolved states must be an isometry on the Bloch sphere. Furthermore, this map leaves the energy eigenvectors invariant, so it must be a rotation around the axis passing through the poles. Thus it can be easily seen, both by equations (41) and (43) and this geometric picture, that the states that

travel the fastest are the ones lying on the equator. For these, $\sin\theta = 1$, and

$$\frac{ds}{dt} = \frac{|\lambda_+ - \lambda_-|}{\hbar}. \tag{44}$$

Hence, if the distance between two states on the Bloch sphere is α, the minimum evolution time to take one into the other is given by

$$\tau = \frac{\hbar\alpha}{|\lambda_+ - \lambda_-|}, \tag{45}$$

a result that agrees with the one obtained by Carlini *et al.* [20] using variational methods. These tend to be more complicated but can however be generalised to impose different restraints on the Hamiltonian depending on the physical situation.

3.2   Using $\mathcal{PT}$ symmetry

In 2007, Bender *et al.* approached this problem through the framework of $\mathcal{PT}$-symmetric Hamiltonians [3]. We can still work in the two-dimensional subspace spanned by the two states, but we have to define the action of $\mathcal{P}$ and $\mathcal{T}$ in a finite-dimensional Hilbert space where spatial coordinates are not explicit. We make use of the basic properties of $\mathcal{P}$ and $\mathcal{T}$, which must still be satisfied: $\mathcal{P}$ is linear, $\mathcal{T}$ is antilinear, $\mathcal{P}^2 = \mathcal{T}^2 = 1$ and $[\mathcal{P}, \mathcal{T}] = 0$. The simplest possibility [23] is then to define $\mathcal{T}$ to perform complex conjugation and

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{46}$$

With these definitions, in order to find the most general 2×2 $\mathcal{PT}$-symmetric

Hamiltonian H, we write

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow (\mathcal{PT})H(\mathcal{PT}) = \begin{pmatrix} d^* & c^* \\ b^* & a^* \end{pmatrix}, \tag{47}$$

so $H = (\mathcal{PT})H(\mathcal{PT})$ implies that H can be written

$$H = \begin{pmatrix} re^{i\beta} & se^{i\delta} \\ se^{-i\delta} & re^{-i\beta} \end{pmatrix}, \tag{48}$$

where r, s, β and δ are real parameters. To simplify calculations, we also require that H is symmetric, that is, we set $\delta = 0$. Since we will need to work with this Hamiltonian both here and in section 4.2, we now proceed to analyse it completely.

The energy eigenvalues are given by $\lambda_{\pm} = r\cos\beta \pm \sqrt{s^2 - r^2\sin^2\beta}$, with the reality condition reading $s^2 > r^2\sin\beta$. As we will verify later, when this condition is satisfied the eigenvalues of H are also eigenvalues of $\mathcal{PT}$, i.e., the system has unbroken $\mathcal{PT}$ symmetry. Hence, we can define a real parameter $\alpha = \arcsin\left(\frac{r}{s}\sin\beta\right)$, in terms of which the difference between eigenvalues may be written $|\lambda_+ - \lambda_-| = 2s\cos\alpha$. The condition for a column vector $\begin{pmatrix} a \\ b \end{pmatrix}$ to be an eigenvector of H is

$$\begin{aligned} \frac{b}{a} &= \frac{1}{s}(-re^{i\beta} + r\cos\beta \pm \sqrt{s^2 - r^2\sin^2\beta}) \\ &= -i\sin\alpha \pm \cos\alpha = \pm e^{\mp i\alpha}, \end{aligned} \tag{49}$$

so we can write the eigenvectors as

$$|E_+\rangle = k_+ e^{i\gamma_+} \begin{pmatrix} 1 \\ e^{-i\alpha} \end{pmatrix}, \quad |E_-\rangle = k_- e^{i\gamma_-} \begin{pmatrix} 1 \\ -e^{i\alpha} \end{pmatrix}, \tag{50}$$

where $k_{\pm}$ and $\gamma_{\pm}$ are real constants. The next step, as outlined in section 2.2, is to

check that these are also eigenvectors of $\mathcal{PT}$, and to ensure that they have eigenvalue 1. Then,

$$\mathcal{PT}|E_+\rangle = k_+ e^{-i\gamma_+} \begin{pmatrix} e^{i\alpha} \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{PT}|E_-\rangle = k_- e^{-i\gamma_-} \begin{pmatrix} -e^{-i\alpha} \\ 1 \end{pmatrix} \quad (51)$$

imply that $\gamma_+ = \alpha/2$ and $\gamma_- = (\pi - \alpha)/2$. We now need to normalise the states so that their $\mathcal{PT}$ norm is ± 1:

$$(\mathcal{PT}|E_+\rangle) \cdot |E_+\rangle = 2k_+^2 \cos\alpha, \quad (\mathcal{PT}|E_-\rangle) \cdot |E_-\rangle = -2k_-^2 \cos\alpha, \quad (52)$$

so we have to define $k_\pm = (2\cos\alpha)^{-1/2}$. Note that $|E_+\rangle$ has $\mathcal{PT}$ norm 1, but $|E_-\rangle$ has norm -1. Finally, the properly normalised eigenvectors can be written

$$|E_+\rangle = \frac{1}{\sqrt{2\cos\alpha}} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix}, \quad |E_-\rangle = \frac{1}{\sqrt{2\cos\alpha}} \begin{pmatrix} ie^{-i\alpha/2} \\ -ie^{i\alpha/2} \end{pmatrix} \quad (53)$$

and $\mathcal{C}$ can be calculated:

$$\mathcal{C} = \frac{1}{2\cos\alpha} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & e^{-i\alpha/2} \end{pmatrix} + \frac{1}{2\cos\alpha} \begin{pmatrix} ie^{-i\alpha/2} \\ -ie^{i\alpha/2} \end{pmatrix} \begin{pmatrix} ie^{-i\alpha/2} & -ie^{i\alpha/2} \end{pmatrix}$$

$$= \frac{1}{\cos\alpha} \begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}. \quad (54)$$

To understand this Hamiltonian fully, we should still calculate the time-evolution operator it creates, as well as its Hermitian conjugate. We then use

$$e^{k\mathbf{1}+i\boldsymbol{\sigma}\cdot\mathbf{a}} = e^k \left(\cos a \mathbf{1} + i\frac{\sin a}{a} \boldsymbol{\sigma} \cdot \mathbf{a} \right), \quad (55)$$

where $a = |\mathbf{a}|$ and $\boldsymbol{\sigma}$ is the vector of the Pauli matrices,

$$
\boldsymbol{\sigma} = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right],
\tag{56}
$$

with

$$
H = r \cos \beta \mathbf{1} + \boldsymbol{\sigma} \cdot (s, 0, ir \sin \beta), \quad H^\dagger = r \cos \beta \mathbf{1} + \boldsymbol{\sigma} \cdot (s, 0, -ir \sin \beta),
\tag{57}
$$

to calculate

$$
\begin{aligned}
e^{-iHt/\hbar} &= \frac{e^{-itr \cos \beta/\hbar}}{\cos \alpha} \begin{pmatrix} \cos(\frac{t}{\hbar}\sqrt{s^2 - r^2 \sin^2 \beta} - \alpha) & -i \sin(\frac{t}{\hbar}\sqrt{s^2 - r^2 \sin^2 \beta}) \\ -i \sin(\frac{t}{\hbar}\sqrt{s^2 - r^2 \sin^2 \beta}) & \cos(\frac{t}{\hbar}\sqrt{s^2 - r^2 \sin^2 \beta} + \alpha) \end{pmatrix} \\
&= \frac{e^{-itr \cos \beta/\hbar}}{\cos \alpha} \begin{pmatrix} \cos(\omega t - \alpha) & -i \sin(\omega t) \\ -i \sin(\omega t) & \cos(\omega t + \alpha) \end{pmatrix},
\end{aligned}
\tag{58}
$$

$$
e^{iH^\dagger t/\hbar} = \frac{e^{itr \cos \beta/\hbar}}{\cos \alpha} \begin{pmatrix} \cos(\omega t - \alpha) & i \sin(\omega t) \\ i \sin(\omega t) & \cos(\omega t + \alpha) \end{pmatrix},
\tag{59}
$$

where we have defined $\omega = \frac{1}{\hbar}\sqrt{s^2 - r^2 \sin^2 \beta} = \frac{1}{2\hbar}|\lambda_+ - \lambda_-|$.

We can now look at what this means for the quantum brachistochrone problem. For simplicity, we can take

$$
|\psi_I\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi_F\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

$$
e^{-iHt/\hbar}|\psi_I\rangle = \frac{e^{-itr \cos \beta/\hbar}}{\cos \alpha} \begin{pmatrix} \cos(\omega t - \alpha) \\ -i \sin(\omega t) \end{pmatrix}.
\tag{60}
$$

We find that the evolution time from $|\psi_I\rangle$ to $|\psi_F\rangle$, before subject to the limit $t \geq$

$\hbar\pi|\lambda_+ - \lambda_-|^{-1}$, is now equal to $t = \hbar(2\alpha + \pi)|\lambda_+ - \lambda_-|^{-1}$. Since we can adjust the parameters of the Hamiltonian to make α arbitrarily close to $-\pi/2$ while keeping $|\lambda_+ - \lambda_-|$ constant, t becomes arbitrarily small!

What happened? Have we exceeded the speed limit imposed by the Anandan-Aharonov relation through the use of non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians? Mostafazadeh has shown [24], using the equivalent Hermitian Hamiltonian, that in fact all that we have done is to change the metric on the Bloch sphere. Indeed we have not violated the speed limit, but merely chose a metric such that the two states are arbitrarily close.

The use of $\mathcal{PT}$-symmetric Hamiltonians has then changed the solution of the quantum brachistochrone to an evolution that can take place in an arbitrarily short time. This happens because the inner product is determined by the Hamiltonian, so we just had to find a Hamiltonian corresponding to an inner product that sees the two states as being very close together.

4   State Discrimination

If we know the state of a quantum system, we can predict the probabilities associated to the possible outcomes of any measurement. However, the state itself is not an observable, so it cannot be directly determined by observation. This is an essential fact in quantum information, since in any computational process, after the processing occurs, the processed state has to be determined in order to read out the desired information. However, we usually know what the set of possible final states is, and we just have to discriminate between the states in that set.

Another example to illustrate this problem is given in the context of quantum communications. Suppose that Alice wants to send a message to Bob, and to do that Alice chooses a state $|\psi_i\rangle$ from a given set of states, and sends this state to Bob. Further suppose that the probability of Alice choosing the state $|\psi_i\rangle$ is p_i, and that

this probability and the set of possible states are also known by Bob. Then Bob must apply a measurement to the received state in order to discover which message Alice has sent.

We will limit our analysis to a set of possible final states containing two pure states, since that contains all the important features of the general problem, even though it is the simplest case. It is also the case for which we can use the Bloch sphere, thus finding some analogies with the quantum brachistochrone. In the case of conventional quantum mechanics, the generalisations to more states or mixed states, as well as experimental results, can be found in the literature [25, 26].

4.1   In conventional quantum mechanics

One usually thinks of a measurement as being associated to an observable property A corresponding to a self-adjoint operator $\hat{A}$, which can be written in a diagonal form $\hat{A} = \sum_i a_i |a_i\rangle\langle a_i|$. For simplicity, we assume that all a_i are distinct and the kets $|a_i\rangle$ form a complete orthonormal set. If a system is in the state $|\psi\rangle$, the probability that a measurement of A will give the outcome a_i is then given by $P(a_i) = |\langle a_i|\psi\rangle|^2$.

We can think of measurements somewhat differently by noting that $P(a_i)$ is equal to $\langle |a_i\rangle\langle a_i| \rangle$ (where $\langle \hat{O} \rangle$, the *expectation value* of $\hat{O}$, is defined as $\langle\psi|\hat{O}|\psi\rangle$). A measurement then corresponds to a set of projection operators $\{|a_i\rangle\langle a_i|\}$ onto the elements of a complete and orthonormal set of vectors.

Note that the completeness of the set of kets $\{|a_i\rangle\}$ as a basis for the Hilbert space can be expressed as the condition on the set of projection operators $\{|a_i\rangle\langle a_i|\}$, $\sum_i |a_i\rangle\langle a_i| = \mathbf{1}$. The same goes for orthonormality, since the condition on the vectors $\langle a_i|a_j\rangle = \delta_{ij}$ can be expressed as the condition $|a_i\rangle\langle a_i|a_j\rangle\langle a_j| = \delta_{ij}|a_i\rangle\langle a_i|$ on the projection operators. If a general set of operators satisfies these conditions, we also say it is, respectively, complete and orthonormal.

The other two key properties of projection operators we are using are Hermiticity

and positivity, which guarantee that $\langle |a_i\rangle\langle a_i| \rangle$ is a positive real number and can therefore be interpreted probabilistically (completeness, together with these properties, ensures it is not larger than 1).

The theory of *generalised measurements* is set up by dropping the requirement of orthogonality, that is, by considering all sets of Hermitian and positive operators π_i that are complete. Any set that satisfies these properties is called a *positive operator measure*, or POM. In this theory, each operator π_i then corresponds to one possible outcome occuring with probability $\langle \pi_i \rangle$.

Any POM corresponds to the effect of a measurement on the combination of the original system with an auxiliary system (the combination is described by the tensor product the two Hilbert spaces). This extension of the system is known as *Naĭmark extension* [27]. Remarkably, the converse is also true, that is, any measurement made on the system coupled to another system corresponds to a generalised measurement described by a POM. The study of the possible measurements we can do on a quantum system thus reduces to the study of all possible POMs.

We now outline two possible strategies if we have a system that we know is in one of two states, $|\psi_1\rangle$ and $|\psi_2\rangle$, with associated probabilities p_1 and $p_2 = 1 - p_1$. Let 2ϵ be the distance between the states in the Bloch sphere. For simplicity, we can choose coordinates such that $|\psi_1\rangle$ is at $(\epsilon, 0)$ and ψ_2 at $(-\epsilon, 0)$, so they can be written

$$|\psi_1\rangle = \begin{pmatrix} \cos\frac{\epsilon}{2} \\ \sin\frac{\epsilon}{2} \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} \cos\frac{\epsilon}{2} \\ -\sin\frac{\epsilon}{2} \end{pmatrix}. \tag{61}$$

We will see that perfect unambiguous discrimination is in general impossible to achieve, and the best that can be done is to optimise the measurement strategy with respect to some criteria. This fact, besides the obvious problem that it represents in the contexts mentioned above, can also be used to one's advantage in quantum criptography [28].

4.1.1   Minimum error discrimination

The first strategy, called *minimum error discrimination*, allows for two possible outcomes, 1 and 2, associated with the operators π_1 and π_2 (remember that for $\{\pi_1, \pi_2\}$ to form a POM, we must have $\pi_1 + \pi_2 = \mathbf{1}$). The outcome i then tells us that the system is in the state $|\psi_i\rangle$, and we wish to minimize the probability of making an error in identifying the correct state of the system.

This probability is given by

$$
\begin{aligned}
P_{\text{error}} &= P(\psi_1)P(2|\psi_1) + P(\psi_2)P(1|\psi_2) \\
&= p_1\langle\psi_1|\pi_2|\psi_1\rangle + p_2\langle\psi_2|\pi_1|\psi_2\rangle \\
&= p_1 - \text{tr}[(p_1|\psi_1\rangle\langle\psi_1| - p_2|\psi_2\rangle\langle\psi_2|)\pi_1],
\end{aligned}
\tag{62}
$$

where in the last line we used $\pi_2 = \mathbf{1} - \pi_1$ and the fact that the states are normalised. The minimum probability of error then occurs when the trace in the second term is maximized. This trace is equal to $a\lambda_+ + b\lambda_-$, where $\lambda_\pm$ are the eigenvalues of $p_1|\psi_1\rangle\langle\psi_1| - p_2|\psi_2\rangle\langle\psi_2|$ and a and b are real numbers that we can adjust between 0 and 1, by the properties of π_1.

To find the eigenvalues of $p_1|\psi_1\rangle\langle\psi_1| - p_2|\psi_2\rangle\langle\psi_2|$, we write it in matrix form:

$$
p_1|\psi_1\rangle\langle\psi_1| - p_2|\psi_2\rangle\langle\psi_2| = \begin{pmatrix} (p_1 - p_2)\cos^2\frac{\epsilon}{2} & (p_1 + p_2)\cos\frac{\epsilon}{2}\sin\frac{\epsilon}{2} \\ (p_1 + p_2)\cos\frac{\epsilon}{2}\sin\frac{\epsilon}{2} & (p_1 - p_2)\sin^2\frac{\epsilon}{2} \end{pmatrix}.
\tag{63}
$$

Its eigenvalues are

$$
\lambda_\pm = \frac{1}{2}\left(p_1 - p_2 \pm \sqrt{(p_1 - p_2)^2 + 4p_1 p_2 \sin^2\epsilon} \right),
\tag{64}
$$

from which we can easily see that λ_+ is positive and λ_- negative. This means that the trace above is maximized when π_1 projects onto the eigenvector corresponding

to λ_+, that is, when $a = 1$ and $b = 0$. The minimum probability of error is then

$$P_{\text{err min}} = p_1 - \lambda_+ = \frac{1}{2}(p_1 + p_2 - \sqrt{(p_1 + p_2)^2 - 4p_1 p_2 \cos^2 \epsilon})$$

$$= \frac{1}{2}(1 - \sqrt{1 - 4p_1 p_2 \cos^2 \epsilon}) = \frac{1}{2}(1 - \sqrt{1 - 4p_1 p_2 |\langle \psi_1 | \psi_2 \rangle|^2}), \qquad (65)$$

which is called the *Helstrom bound*. Note that the Helstrom bound goes to zero if the states are orthogonal or if p_1 or p_2 are zero, that is, if we know what the state is from the beginning.

This type of strategy can be generalised to minimise a more general cost function, or to maximise the mutual information gained [29, 30, 31]. The strategy is easy to generalise to mixed states, since we just need to substitute the projection operators $|\psi_i\rangle\langle\psi_i|$ by density operators. If we increase the number of possible states, however, the calculations quickly get quite complicated.

4.1.2   Unambiguous discrimination

The other strategy we will present is called *unambiguous discrimination*, and can be used if an error in identifying the state is not allowed. The price of not allowing for errors, as we will see, is that we generally have to allow for a third outcome, which does not allow us to take any conclusion about the system.

We then want our POM to have two operators, π_1 and π_2, where π_i corresponds to an outcome that allows us to unambiguously identify the state of the system as $|\psi_i\rangle$. This can be done by requiring $\pi_1|\psi_2\rangle = 0$ and $\pi_2|\psi_1\rangle = 0$, which guarantees that outcome 1 never happens if the state is $|\psi_2\rangle$ and vice versa. Using the coordinates

chosen in (61), we can write such operators as

$$\pi_1 = a_1 \begin{pmatrix} \sin\frac{\epsilon}{2} \\ \cos\frac{\epsilon}{2} \end{pmatrix} \begin{pmatrix} \sin\frac{\epsilon}{2} & \cos\frac{\epsilon}{2} \end{pmatrix} = a_1 \begin{pmatrix} \sin^2\frac{\epsilon}{2} & \cos\frac{\epsilon}{2}\sin\frac{\epsilon}{2} \\ \cos\frac{\epsilon}{2}\sin\frac{\epsilon}{2} & \cos^2\frac{\epsilon}{2} \end{pmatrix},$$

$$\pi_2 = a_2 \begin{pmatrix} \sin\frac{\epsilon}{2} \\ -\cos\frac{\epsilon}{2} \end{pmatrix} \begin{pmatrix} \sin\frac{\epsilon}{2} & -\cos\frac{\epsilon}{2} \end{pmatrix} = a_2 \begin{pmatrix} \sin^2\frac{\epsilon}{2} & -\cos\frac{\epsilon}{2}\sin\frac{\epsilon}{2} \\ -\cos\frac{\epsilon}{2}\sin\frac{\epsilon}{2} & \cos^2\frac{\epsilon}{2} \end{pmatrix} \tag{66}$$

with $0 \leq a_1, a_2 \leq 1$. Note that $\pi_1 + \pi_2 = \mathbf{1}$ is only possible if $\epsilon = \frac{\pi}{2}$, which corresponds to the case where the two possible states are orthogonal.

We then need another operator, $\pi_0 = \mathbf{1}-\pi_1-\pi_2$, corresponding to an inconclusive outcome. The probability of this outcome is then given by

$$P_0 = P(\psi_1)P(0|\psi_1) + P(\psi_2)P(0|\psi_2)$$

$$= p_1\langle\psi_1|\pi_0|\psi_1\rangle + p_2\langle\psi_2|\pi_0|\psi_2\rangle$$

$$= 1 - (p_1\langle\psi_1|\pi_1|\psi_1\rangle + p_2\langle\psi_2|\pi_2|\psi_2\rangle)$$

$$= 1 - \sin^2\epsilon(a_1p_1 + a_2p_2). \tag{67}$$

We would like to minimise this probability, which is equivalent to maximising $a_1p_1 + a_2p_2$ subject to the constraint that π_0 remains a positive operator. Since π_0 can be written

$$\pi_0 = \begin{pmatrix} 1 - (a_1 + a_2)\sin^2\frac{\epsilon}{2} & (a_2 - a_1)\cos\frac{\epsilon}{2}\sin\frac{\epsilon}{2} \\ (a_2 - a_1)\cos\frac{\epsilon}{2}\sin\frac{\epsilon}{2} & 1 - (a_1 + a_2)\cos^2\frac{\epsilon}{2} \end{pmatrix}, \tag{68}$$

the condition that its eigenvalues are nonnegative is

$$1 - a_1 - a_2 + a_1a_2\sin^2\epsilon \geq 0. \tag{69}$$

The variational problem can then be solved, giving the optimal coefficients

$$a_1 = \frac{1 - \sqrt{p_2/p_1}\cos\epsilon}{\sin^2\epsilon}, \quad a_2 = \frac{1 - \sqrt{p_1/p_2}\cos\epsilon}{\sin^2\epsilon} \tag{70}$$

and $P_0 = 2\sqrt{p_0 p_1}\cos\epsilon$. Note that this probability is very large if the states are very close to each other.

4.2 Using $\mathcal{PT}$ symmetry

4.2.1 An inner product for which the 2 states are orthogonal

In section 3.2, we made use of the fact that the inner product is determined by the Hamiltonian to make the distance between two states arbitrarily small. Can we now use the same technique to separate the possible final states if they are very close to each other in the usual Hilbert space? In fact, for any two given states, can we find a Hamiltonian that makes them orthogonal?

Once more, we can work in the two-dimensional subspace spanned by the two states $|\psi_1\rangle$ and $|\psi_2\rangle$. Let 2ϵ be the distance between the two states in the Bloch sphere. We can reparametrise the sphere so that both states lie along the same meridian, i.e., $|\psi_1\rangle$ corresponds to (θ, φ), while $|\psi_2\rangle$ is at $(\theta+2\epsilon, \varphi)$. This is equivalent to writing

$$|\psi_1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} \cos\frac{\theta+2\epsilon}{2} \\ \sin\frac{\theta+2\epsilon}{2}e^{i\varphi} \end{pmatrix}. \tag{71}$$

Note that we still have the freedom to choose the parametrisation more strictly, by choosing specific values for θ and φ. In fact we will do so later, but before we will check which choice would be the most convenient.

Recovering equations (46)-(54), $\mathcal{T}$ performs complex conjugation,

$$
\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} re^{i\beta} & s \\ s & re^{-i\beta} \end{pmatrix}, \quad \mathcal{C} = \frac{1}{\cos\alpha} \begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}, \quad (72)
$$

where r, s and β are real and $\alpha = \arcsin\left(\frac{r}{s}\sin\beta\right)$, which is real because we are in the region of unbroken $\mathcal{PT}$ symmetry when $s^2 > r^2 \sin^2\beta$. We now calculate

$$
\mathcal{CP} = \frac{1}{\cos\alpha} \begin{pmatrix} 1 & i\sin\alpha \\ -i\sin\alpha & 1 \end{pmatrix}. \tag{73}
$$

With that, we can form the bras corresponding to the kets above:

$$
\langle\psi_1| = \frac{1}{\cos\alpha}\left(\cos\tfrac{\theta}{2} + i\sin\alpha\sin\tfrac{\theta}{2}e^{-i\varphi} \quad -i\sin\alpha\cos\tfrac{\theta}{2} + \sin\tfrac{\theta}{2}e^{-i\varphi}\right) \tag{74}
$$

and

$$
\begin{aligned}
\langle\psi_1|\psi_1\rangle &= \frac{1}{\cos\alpha}\left(\cos^2\frac{\theta}{2} + i\sin\alpha\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\varphi} - i\sin\alpha\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} + \sin^2\frac{\theta}{2}\right) \\
&= \frac{1}{\cos\alpha}(1 + \sin\alpha\sin\theta\sin\varphi),
\end{aligned} \tag{75}
$$

and the same for $|\psi_2\rangle$ with the appropriate substitution of θ for $\theta+2\epsilon$. We choose not to normalise the states, for now, since that makes no difference to their orthogonality.

The inner product then gives

$$
\begin{aligned}
\langle\psi_1|\psi_2\rangle &= \frac{1}{\cos\alpha}\times \\
&\left[\left(\cos\frac{\theta}{2} + i\sin\alpha\sin\frac{\theta}{2}e^{-i\varphi}\right)\cos\frac{\theta+2\epsilon}{2} + \left(-i\sin\alpha\cos\frac{\theta}{2} + \sin\frac{\theta}{2}e^{-i\varphi}\right)\sin\frac{\theta+2\epsilon}{2}e^{i\varphi}\right] \\
&= \frac{1}{\cos\alpha}\left[i\sin\alpha\left(\sin\frac{\theta}{2}\cos\frac{\theta+2\epsilon}{2}e^{-i\varphi} - \cos\frac{\theta}{2}\sin\frac{\theta+2\epsilon}{2}e^{i\varphi}\right) + \cos\epsilon\right],
\end{aligned} \tag{76}
$$

which means that the orthogonality condition reads

$$
\begin{aligned}
i \sin \alpha &= \frac{\cos \epsilon}{\cos \frac{\theta}{2} \sin \frac{\theta+2\epsilon}{2} e^{i\varphi} - \sin \frac{\theta}{2} \cos \frac{\theta+2\epsilon}{2} e^{-i\varphi}} \\
&= \frac{\cos \epsilon}{\sin \epsilon \cos \varphi + i \sin(\theta + \epsilon) \sin \varphi}.
\end{aligned} \tag{77}
$$

In order for this equation to be solvable in α, we have, as anticipated, to be more specific in our reparametrisation of the Bloch sphere. First, since the left-hand side is purely imaginary, the right-hand side must also be purely imaginary. This implies that $\cos \varphi = 0$, which in turn implies that $\sin \varphi = \pm 1$. For definiteness, we choose the minus sign, or $\varphi = -\pi/2$, obtaining

$$
\sin \alpha = \frac{\cos \epsilon}{\sin(\theta + \epsilon)}, \tag{78}
$$

which means that there exists an α that makes the two states orthogonal if

$$
|\sin(\theta + \epsilon)| > |\cos \epsilon| \Leftrightarrow \frac{\pi}{2} - 2\epsilon < \theta < \frac{\pi}{2}, \tag{79}
$$

another condition that we need to impose on our parametrisation. Note that this condition is impossible to satisfy, as it should be, if $2\epsilon = 0 \Leftrightarrow |\psi_1\rangle = |\psi_2\rangle$. To simplify calculations, we can pick $\theta = \frac{\pi}{2} - \epsilon$, or $\sin \alpha = \cos \epsilon$. Hence, in order to completely distinguish the two states, all that needs to be done is to construct the projection operators that leave one state invariant and annihilate the other. However, we need to normalise the states before doing calculating the projection operators. From equation (75) with the appropriate substitutions, we obtain

$$
\langle \psi_1 | \psi_1 \rangle = \frac{1}{\sin \epsilon}(1 - \cos^2 \epsilon) = \sin \epsilon = \langle \psi_2 | \psi_2 \rangle, \tag{80}
$$

with the last equality following from the fact the norm of $|\psi_2\rangle$ had the same form with the replacement $\sin \theta \to \sin(\theta + 2\epsilon)$, and with $\theta = \frac{\pi}{2} - \epsilon$, $\sin(\theta + 2\epsilon) = \sin \theta$.

Hence, after reparametrising and normalising,

$$|\psi_1\rangle = \frac{1}{\sqrt{\sin\epsilon}} \begin{pmatrix} \cos\frac{\pi-2\epsilon}{4} \\ -i\sin\frac{\pi-2\epsilon}{4} \end{pmatrix},$$

$$\langle\psi_1| = \frac{1}{\sin^{3/2}\epsilon} \left(\cos\frac{\pi-2\epsilon}{4} - \cos\epsilon\sin\frac{\pi-2\epsilon}{4} \quad -i\cos\epsilon\cos\frac{\pi-2\epsilon}{4} + i\sin\frac{\pi-2\epsilon}{4} \right), \qquad (81)$$

and the same holds for $|\psi_2\rangle$ with the substitution of $\pi-2\epsilon$ by $\pi+2\epsilon$. We can finally write the projection operators:

$$|\psi_1\rangle\langle\psi_1| =$$

$$\frac{1}{\sin^2\epsilon} \begin{pmatrix} \cos^2\frac{\pi-2\epsilon}{4} - \cos\epsilon\sin\frac{\pi-2\epsilon}{4}\cos\frac{\pi-2\epsilon}{4} & -i\cos\epsilon\cos^2\frac{\pi-2\epsilon}{4} + i\sin\frac{\pi-2\epsilon}{4}\cos\frac{\pi-2\epsilon}{4} \\ -i\cos\frac{\pi-2\epsilon}{4}\sin\frac{\pi-2\epsilon}{4} + i\cos\epsilon\sin^2\frac{\pi-2\epsilon}{4} & -\cos\epsilon\cos\frac{\pi-2\epsilon}{4}\sin\frac{\pi-2\epsilon}{4} + \sin^2\frac{\pi-2\epsilon}{4} \end{pmatrix}$$

$$=\frac{1}{2\sin^2\epsilon} \begin{pmatrix} 1+\cos\frac{\pi-2\epsilon}{2} - \cos\epsilon\sin\frac{\pi-2\epsilon}{2} & -i\cos\epsilon(1+\cos\frac{\pi-2\epsilon}{2}) + i\sin\frac{\pi-2\epsilon}{2} \\ -i\sin\frac{\pi-2\epsilon}{2} + i\cos\epsilon(1-\cos\frac{\pi-2\epsilon}{2}) & -\cos\epsilon\sin\frac{\pi-2\epsilon}{2} + 1 - \cos\frac{\pi-2\epsilon}{2} \end{pmatrix}$$

$$=\frac{1}{2\sin^2\epsilon} \begin{pmatrix} \sin^2\epsilon + \sin\epsilon & -i\cos\epsilon\sin\epsilon \\ -i\cos\epsilon\sin\epsilon & \sin^2\epsilon - \sin\epsilon \end{pmatrix}$$

$$=\frac{1}{2\sin\epsilon} \begin{pmatrix} \sin\epsilon + 1 & -i\cos\epsilon \\ -i\cos\epsilon & \sin\epsilon - 1 \end{pmatrix} \qquad (82)$$

and for $|\psi_2\rangle$ we take it from the second line, swapping $\pi-2\epsilon$ for $\pi+2\epsilon$ to obtain

$$|\psi_2\rangle\langle\psi_2| =$$

$$\frac{1}{2\sin^2\epsilon} \begin{pmatrix} 1+\cos\frac{\pi+2\epsilon}{2} - \cos\epsilon\sin\frac{\pi+2\epsilon}{2} & -i\cos\epsilon(1+\cos\frac{\pi+2\epsilon}{2}) + i\sin\frac{\pi+2\epsilon}{2} \\ -i\sin\frac{\pi-2\epsilon}{2} + i\cos\epsilon(1-\cos\frac{\pi-2\epsilon}{2}) & -\cos\epsilon\sin\frac{\pi+2\epsilon}{2} + 1 - \cos\frac{\pi+2\epsilon}{2} \end{pmatrix}$$

$$=\frac{1}{2\sin^2\epsilon} \begin{pmatrix} \sin^2\epsilon - \sin\epsilon & i\cos\epsilon\sin\epsilon \\ i\cos\epsilon\sin\epsilon & \sin^2\epsilon + \sin\epsilon \end{pmatrix}$$

$$=\frac{1}{2\sin\epsilon} \begin{pmatrix} \sin\epsilon - 1 & i\cos\epsilon \\ i\cos\epsilon & \sin\epsilon + 1 \end{pmatrix}. \qquad (83)$$

Note that we have, as we must, $|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| = \mathbf{1}$. Also, $|\psi_1\rangle\langle\psi_1|$ can be written as

$$|\psi_1\rangle\langle\psi_1| = \frac{1}{2}\mathbf{1} + \boldsymbol{\sigma} \cdot \left(\frac{-i}{2}\cot\epsilon, 0, \frac{1}{\sin\epsilon}\right), \qquad (84)$$

while

$$H = r\cos\beta\mathbf{1} + \boldsymbol{\sigma}\cdot(s, 0, ir\sin\beta) = \sqrt{r^2 - s^2\cos^2\epsilon}\,\mathbf{1} + \boldsymbol{\sigma}\cdot(s, 0, is\cos\epsilon). \qquad (85)$$

Both of these operators can be thought of as something equivalent to applying a magnetic field in a complex direction.

4.2.2 A Hamiltonian under which the states evolve into orthogonal states

Note that this theory is still in its beginning, and it is conceivable that in some cases we may be able to implement a non-Hermitian Hamiltonian, but not a non-Hermitian observable. In the applications of this theory to optics that we briefly mentioned in section 2.4, for example, there is no analogue for other operators besides the Hamiltonian, since there we are not talking about a quantum theory. If such a case is true, we can still use $\mathcal{PT}$ symmetry to simplify state discrimination. All we have to do is to find a Hamiltonian under which the two states evolve into two states that are orthogonal under the conventional inner product, and then proceed to make a measurement in a real direction.

The inner product (now in the *conventional* Hilbert space) of the time-evolved states is given by $\langle\psi_1|e^{iH^\dagger t/\hbar}e^{-iHt/\hbar}|\psi_2\rangle$. We then use equations (55)-(59) to see

$$
\begin{aligned}
e^{iH^\dagger t/\hbar}e^{-iHt/\hbar} &= \frac{1}{\cos^2\alpha}\begin{pmatrix} \cos(\omega t - \alpha) & -i\sin(\omega t) \\ -i\sin(\omega t) & \cos(\omega t + \alpha) \end{pmatrix}\begin{pmatrix} \cos(\omega t - \alpha) & i\sin(\omega t) \\ i\sin(\omega t) & \cos(\omega t + \alpha) \end{pmatrix} \\
&= \frac{1}{\cos^2\alpha}\begin{pmatrix} \cos^2(\omega t - \alpha) + \sin^2(\omega t) & -2i\sin^2(\omega t)\sin\alpha \\ 2i\sin^2(\omega t)\sin\alpha & \cos^2(\omega t + \alpha) + \sin^2(\omega t) \end{pmatrix}, \quad (86)
\end{aligned}
$$

where $\omega = \frac{1}{\hbar}\sqrt{s^2 - r^2 \sin^2 \beta}$. Note that in the Hermitian limit $\alpha \to 0$, this becomes the identity.

Using the same coordinates chosen in (71), since the convenient choice of θ and φ might be different in this case, the inner product evolves with time as

$$
\begin{aligned}
\langle \psi_1, t | \psi_2, t \rangle &= \langle \psi_1 | e^{iH^\dagger t/\hbar} e^{-iHt/\hbar} | \psi_2 \rangle \\
&= \cos\frac{\theta}{2} \cos\frac{\theta + 2\epsilon}{2} \left(\frac{1 + \cos(2\omega t)\cos(2\alpha) + \sin(2\omega t)\sin(2\alpha)}{2} + \sin^2(\omega t) \right) \\
&\quad + 2i \sin\frac{\theta}{2} \cos\frac{\theta + 2\epsilon}{2} \sin^2(\omega t) \sin\alpha(\cos\varphi - i\sin\varphi) \\
&\quad - 2i \cos\frac{\theta}{2} \sin\frac{\theta + 2\epsilon}{2} \sin^2(\omega t) \sin\alpha(\cos\varphi + i\sin\varphi) \\
&\quad + \sin\frac{\theta}{2} \sin\frac{\theta + 2\epsilon}{2} \left(\frac{1 + \cos(2\omega t)\cos(2\alpha) - \sin(2\omega t)\sin(2\alpha)}{2} + \sin^2(\omega t) \right) \\
&= \frac{1}{2} \cos\epsilon[1 + \cos(2\omega t)\cos(2\alpha) + 2\sin^2(\omega t)] + \frac{1}{2} \cos(\theta + \epsilon)\sin(2\omega t)\sin(2\alpha) \\
&\quad + 2\sin^2(\omega t)\sin\alpha\sin(\theta + \epsilon)\sin\varphi - 2i\sin^2(\omega t)\sin\alpha\sin\epsilon\cos\varphi. \quad (87)
\end{aligned}
$$

All terms are real, except the last one. We can make that vanish by choosing, as we did in the previous section, $\cos\varphi = 0$. To simplify calculations, we also maintain the choices $\sin\varphi = -1$ and $\theta = \frac{\pi}{2} - \epsilon$, obtaining

$$
\begin{aligned}
\langle \psi_1, t | \psi_2, t \rangle &= \frac{1}{2} \cos\epsilon[1 + (1 - 2\sin^2(\omega t))(1 - 2\sin^2\alpha) + 2\sin^2(\omega t)] - 2\sin^2(\omega t)\sin\alpha \\
&= \cos\epsilon[\cos^2\alpha + 2\sin^2(\omega t)\sin^2\alpha] - 2\sin^2(\omega t)\sin\alpha, \quad (88)
\end{aligned}
$$

which vanishes when

$$
\sin^2(\omega t) = \frac{1}{2} \frac{\cos^2\alpha\cos\epsilon}{\sin\alpha - \sin^2\alpha\cos\epsilon}, \quad (89)
$$

which has a solution in t if $\epsilon \neq 0$. Note that, in a sort of *dejà vu* of the quantum brachistochrone, the time needed for this evolution is arbitrarily small and goes to 0 as $\cos\alpha \to 0$, or $\alpha \to \pm\frac{\pi}{2}$.

5   Conclusions

In this dissertation, we have considered two simple problems in quantum mechanics. They can both be formulated in a two-dimensional Hilbert space, and we have considered the limitations imposed on both situations when using the canonical postulates of quantum mechanics. However, if we relax the requirement of Hermiticity on the Hamiltonian and consider $\mathcal{PT}$-symmetric Hamiltonians instead, we have seen that the limitations seem to disappear instantly. This is caused by the fact that, in the $\mathcal{PT}$-symmetric theory, the inner product on the Hilbert space is determined by the choice of Hamiltonian. This contrasts with the conventional theory, where the conventional inner product is postulated beforehand.

Though this theory is mathematically consistent and we can interpret its results physically, no one has yet been able to test its predictions by implementing a non-Hermitian Hamiltonian in a quantum-mechanical context. This implies that the key mechanism by which the Hamiltonian might tell the system to switch Hilbert spaces still remains mysterious.

This switch would probably be one of the most intriguing features to study when a way to construct a $\mathcal{PT}$-symmetric Hamiltonian experimentally is finally conceived. Until then, the world of new possibilities brought by this generalisation will surely continue to be explored.

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